Dynamic Proofs of Euclidean Theorems
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its fortunes. For years he alluded to the misfortune that the management of the journal could not remain in the South, but little by little the realization that "a good and knowing Providence guided my way" grew upon him. He was deeply grateful for the very great personal sacrifices that were made by Professor James and his family over a period of twelve years, to finance and improve the journal [2, 3], and for the financial support that the MAA began to contribute in 1959 by an appropriation from the Jacob Houck Memorial Fund. The journal became an official publication of the MAA with volume XXXV (1962) [4].

In his retirement in Mobile, Professor Sanders wrote "Unstable Foundations of a Marxist State," a philosophical probing in a lucid style reminiscent of some of the better features of Spinoza's "geometrical" argumentation. This book-length manuscript was finished at a time when a "soft line" on Communism was becoming fashionable and the work remains unpublished.

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4. R. E. Horton, Editorials facing pp. 125, 249, Ibid.

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DYNAMIC PROOFS OF EUCLIDEAN THEOREMS

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Simple observations about transformations of the plane lead to elegant proofs of unusual Euclidean theorems. The theorems are easily stated, and many of them can be conjectured from hand-drawn pictures. The proofs use neither coordinates nor vectors, and the simplest of them require only a slight familiarity with rotations and translations. We look at some of these first, and then introduce similarity transformations to show that a number of theorems which have appeared in the literature as isolated results are really special cases of two or

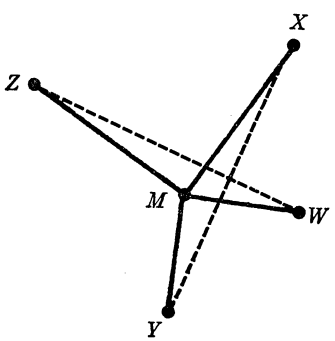

FIG. 1.

three general theorems. A pleasant by-product of the proofs given here is that they yield results about quadrilaterals without assuming them to be either convex or simple (compare, for example, Theorems 1 and 3 with their counterparts in [5]).

The first lemma is about Figure 1.

LEMMA 1. *If isosceles triangles ZMX and YMW have right angles at M, then $\overline{YX}$ and $\overline{ZW}$ are perpendicular and congruent.*

Proof. This is because M_{90}, the counterclockwise rotation of 90 degrees about M, moves Y to W and X to Z.

LEMMA 2. *If Z and X are centers of squares that lie on sides of ABC, built towards the exterior of ABC, and if M is the midpoint of the third side, then ZMX is isosceles and has a right angle at M.*

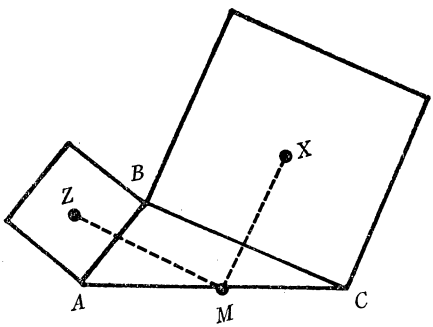

FIG. 2.

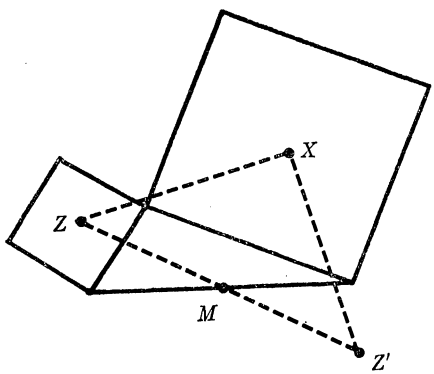

FIG. 3.

To see why, note that the composite $T = M_{180}X_{90}Z_{90}$ of rotations about Z, X and M is a translation. This is because the degree measures of the rotations add up to an integral multiple of 360. Under this composite, A goes first to B, then to C and finally back to A. But the only translation with a fixed point is the identity transformation. Since $T = I$, the successive images of Z are

$$Z_{90}(Z) = Z,$$
$$X_{90}(Z) = \text{some point which we call } Z',$$

and

$$M_{180}(Z') = T(Z) = Z.$$

Our picture now looks like Figure 3. Because $X_{90}(Z) = Z'$, we know that $ZX = XZ'$ and that ZXZ' has a right angle at X. We also know that M is the midpoint of $\overline{ZZ'}$ because $M_{180}(Z') = Z$. Thus M is the midpoint of the hypotenuse of a right isosceles triangle, which makes ZMX isosceles with a right angle at M.

The conclusion of Lemma 2 holds if the squares are constructed towards the interior of the triangle instead of towards the exterior. In the proof, one merely replaces the counterclockwise Z_{90} and X_{90} by the clockwise Z_{-90} and X_{-90}.

THEOREM 1. [1, 3] *If X, Y and Z, W are opposite pairs of centers of squares on the sides of a quadrilateral that lie towards the quadrilateral's exterior, then $\overline{YX}$ and $\overline{WZ}$ are perpendicular and congruent.*

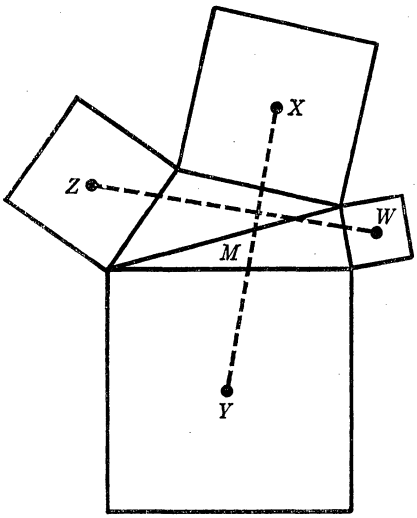

FIG. 4.

Proof. Let M be the midpoint of a diagonal of the quadrilateral, and apply M_{90}.

Theorem 1 holds also if the squares all lie towards the quadrilateral's interior. The proof is the same: M_{90} throws one segment onto the other.

If the quadrilateral happens to be a parallelogram, then $ZXWY$ is a square, because all four of ZMX, XMW, WMY and YMZ are isosceles and have right angles at M.

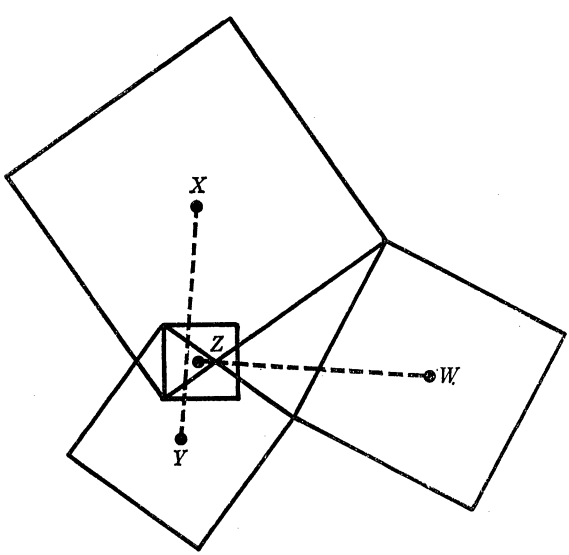

FIG. 5.

It should be emphasized here that the quadrilateral need not be convex, or even simple. To place the squares when the quadrilateral has no obvious interior, one traverses the quadrilateral in one of the two possible directions, laying off squares to the right.

If we shrink an edge of the quadrilateral to a point, we see a theorem about squares on a triangle.

THEOREM 2. [**1, 3**] *If squares are constructed on the sides of a triangle towards the triangle's exterior, then the segment joining two of the centers is perpendicular and congruent to the segment joining the third center to the vertex opposite it.*

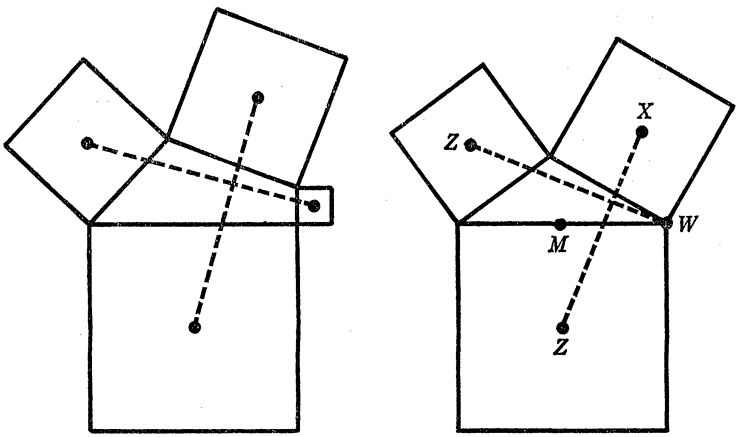

FIG. 6.

Proof. Look at Figure 6, and apply M_{90} as in Lemma 1.

Theorem 2 has the corollary that the lines joining the square centers to the vertices opposite them are concurrent (Figure 7). This is because the lines are altitudes of XYZ.

One of the first theorems of Euclidean geometry is that the midpoints of the sides of a quadrilateral are themselves the vertices of a parallelogram.

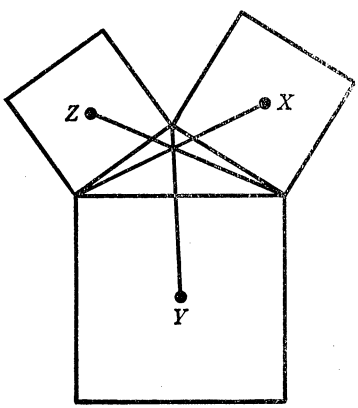

FIG. 7.

THEOREM 3. *The vertices Z, X, W, Y of equilateral triangles built on the sides of a quadrilateral, and lying alternately towards the interior and the exterior of the quadrilateral, are themselves the vertices of a parallelogram.*

Proof. The composite $C_{-60}A_{60}$ is a translation that takes Y to W and Z to X. Thus $\overline{YZ}$ and $\overline{WX}$ are parallel and congruent.

The Euclidean midpoint theorem and Theorem 3 are two of a family of theorems.

THEOREM 4. [2] *If Z, X, W, Y are vertices of similar triangles appropriately arranged on the sides of a quadrilateral, then $ZXWY$ is a parallelogram.*

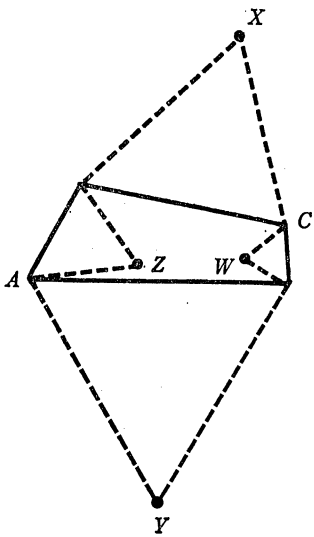

FIG. 8.

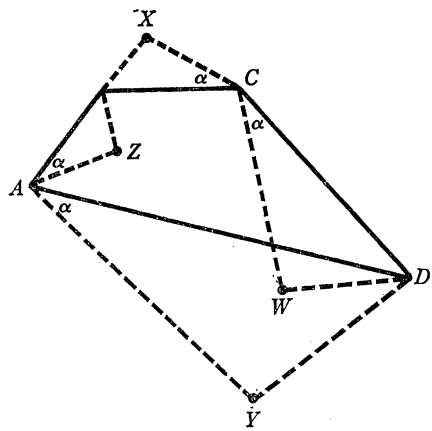

FIG. 9.

Proof. Let α be the measure of the angles that the four similar triangles have at A and C, and let r denote the ratio of AY to AD. Let $A^{1/r}$ denote the central dilatation with center A that multiplies distance by $1/r$. Let C^r denote the central dilatation with center C that multiplies distance by r. Then the composite $C_{-\alpha}C^rA^{1/r}A_\alpha$ is a translation that takes Y to W and Z to X.

If $\alpha=0$ we have a slight generalization of the midpoint theorem. If $r=1$ and $\alpha=60$, we have Theorem 3. Once again, the quadrilateral need not be convex or simple. Shrinking an edge leads to several nice theorems about triangles.

With similarity transformations, one can prove a theorem (Theorem 5) about triangles that has truly surprising corollaries. Among them are the following three:

COROLLARY 1. *Suppose that* 30–60–90 *triangles are built on two sides of an arbitrary triangle towards its exterior, as in Figure* 10. *Let Z and X denote the outer vertices of these triangles, and let M be the midpoint of the remaining side of the given triangle. Then ZMX is equilateral. If, instead, the* 30–60–90 *triangles lie toward the interior of the given triangle, ZMX is still equilateral.*

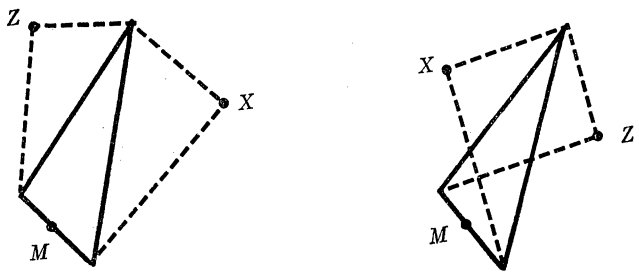

FIG. 10.

COROLLARY 2. (Napoleon's Theorem) *The centers X, Z, M of equilateral triangles constructed on the sides of an arbitrary triangle, and lying towards the triangle's exterior, are themselves the vertices of an equilateral triangle.*

COROLLARY 3. [5, Exercise 23] *Suppose that equilateral triangles are built on the sides of an arbitrary triangle, two towards its exterior and one towards its interior. Let M be the center of the inner one, and Z and X the apexes of the outer ones. Then ZMX is an isosceles triangle, with a* 120° *angle at M.*

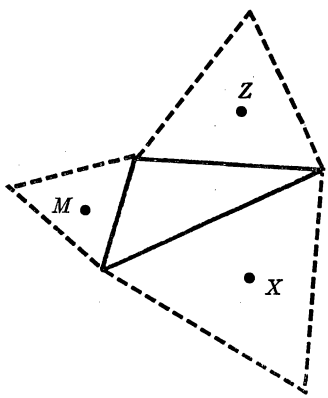

FIG. 11.

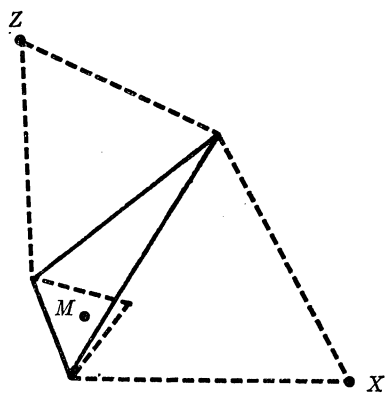

FIG. 12.

In order to state Theorem 5 simply, we assume that angles are "general," that is, that they have a variety of measures which agree modulo 360, and that they are oriented in a counterclockwise fashion. Thus, Figure 13 shows two angles: angle CBA of measure 60, 420, etc., and angle ABC of measure 300, -60, etc.

THEOREM 5. *Let BZC and CXA be nondegenerate similar triangles (with vertices corresponding in the order given) constructed both towards the exterior or both towards the interior of arbitrary triangle ABC. Let the angles BZC and CXA have measure β. Let M be the point in the plane that is equidistant from A and B and that is located so that angle BMA has measure 2β. Then $MZ = MX$.*

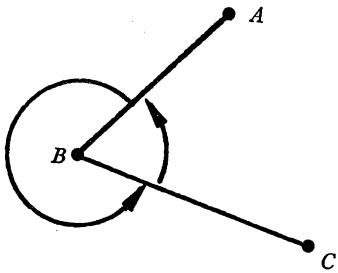

FIG. 13.

Proof. Since BZC and CXA are nondegenerate, $2\beta \neq 360$ and M really does exist. The triangle AMB may be degenerate, however, for if $\beta = 90$ then M is the midpoint of $\overline{AB}$.

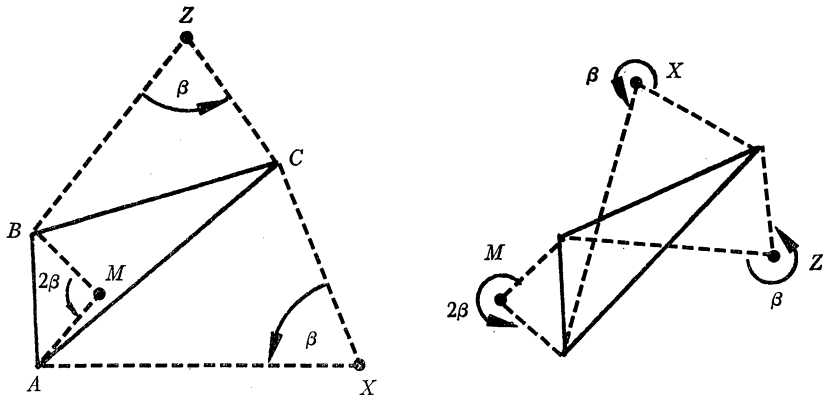

FIG. 14.

Let r denote the ratio ZC/BZ. Then the translation $T = X_\beta X^{1/r} Z^r Z_\beta M_{-2\beta}$ is the identity because it leaves A fixed. If we follow M about, we see that $M_{-2\beta}(M) = M$, that $Z^r Z_\beta(M)$ is some point M', say, and that $X_\beta X^{1/r}(M') = T(M) = M$. Thus $MXM'Z$ is a quadrilateral with $XM' = r \cdot MX$, $ZM' = r \cdot MZ$ and with congruent angles at X and Z. This means that triangles MXM' and MZM' are similar. But they have a common side, so they are congruent and $MZ = MX$.

Proofs of the corollaries. If $\beta = 90$ and $r = \sqrt{3}$, then MXM' and MZM' are 30–60–90 triangles. In particular, $MXM'Z$ has a 60 degree angle at M. This makes ZMX equilateral, which proves Corollary 1.

If $\beta = 120$ and $r = 1$, then Z and X are indistinguishable from M. Consequently $MZ = MX = XZ$, which is Napoleon's theorem.

Corollary 3 is the case $\beta = 60$ and $r = 1$.

If $\beta = -60$ and $r = 1$ we have Corollary 3 with two inner triangles and one outer.

The assignment $\beta = -120$, $r = 1$ produces Napoleon's theorem with three inward triangles.

Lemma 2 is also a corollary of Theorem 5, the squares lying toward the exterior if $\beta = 90$, $r = 1$, and towards the interior of $\beta = -90$, $r = 1$.

We close with a generalization of Lemma 2 that leads to a generalization of Theorem 1.

THEOREM 6. [4] *Suppose that similar isosceles triangles are constructed on the sides of an arbitrary triangle ABC, either both towards the interior of ABC or both towards the exterior. Let X be the apex of one, Z the orthocenter of the other, and let M be the midpoint of the remaining side of ABC. Let β be the measure of the angle at X. Then ZMX has a right angle at M and an angle of measure $\beta/2$ at X.*

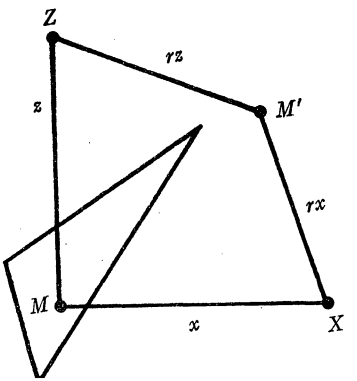

FIG. 15.

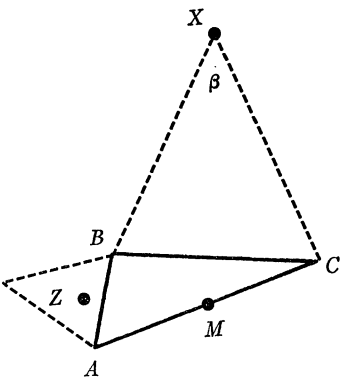

FIG. 16.

The proof is straightforward once one observes that angle AZB has measure $180\text{-}\beta$. Lemma 2 is the case $\beta=90$. If we let h denote the height of X above $\overline{BC}$, then $XM/MZ=BC/2h$. We call this latter number r in the next theorem.

THEOREM 7. *Suppose that similar isosceles triangles lie on the sides of a quadrilateral, all towards the interior or all towards the exterior of the quadrilateral. Let X and Y be apexes of one opposite pair of isosceles triangles, and let Z and W be*

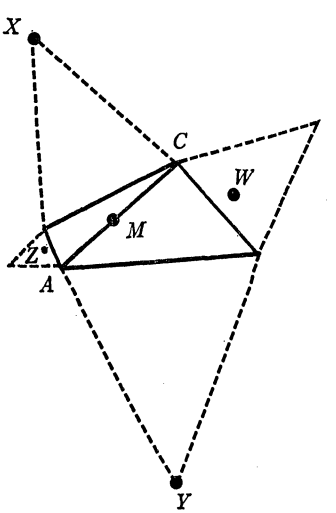

FIG. 17.

the orthocenters of the other two. Then $\overline{YX}$ *and* $\overline{WZ}$ *are perpendicular, and* $YX/WZ = r$.

Proof. Let M be the midpoint of the diagonal $\overline{AC}$. If the triangles lie towards the quadrilateral's exterior, then $M^r M_{90}$ takes X to Z and Y to W. If the triangles lie toward the interior, apply $M^{1/r} M_{90}$ instead.

Again, the quadrilateral need not be convex or simple, and there is an analogous theorem for triangles.

Acknowledgements: I first saw dynamic proofs of Corollaries 1 and 2 in a course that Nicolaas Kuiper gave at the University of Michigan in 1953. Later that year Kenneth Leisenring showed me Theorem 3, and in 1965 he encouraged me to develop dynamic proofs for the African Mathematics Secondary Program. I am grateful to them.

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ANSWERS

A482. For any reals α, β the function

$$f(x) = \left(\sum_{i=1}^{M} x_i^{x-\alpha}\right)\left(\sum_{i=1}^{M} a_i^{\beta-x}\right)$$

is convex and symmetric about $x = 1/2(\alpha+\beta)$, and so increases as x moves away from $x = 1/2(\alpha+\beta)$. With $\alpha = 0$ and $\beta = P$ the given inequality is $f(P) \leqq f(P+1)$.

A483. Expanding out and replacing tan A, tan B and tan C by a, b, and c, respectively we get

$$\frac{a-b}{1+ab} + \frac{b-c}{1+bc} + \frac{c-a}{1+ca} = 0.$$

On combining fractions and factoring we obtain

$$(a-b)(b-c)(c-a) = 0$$

and thus the triangle is isosceles. Note the condition that $A+B+C=\pi$ is not necessary.

A484. Fermat's theorem states that if p is a prime and m is not divisible by p, then $m^{p-1} \equiv 1 \pmod{p}$. The relationship $a^2+b^2=c^2$ may be written as

$$(a^{3-1} - 1) + (b^{3-1} - 1) = c^2 - 2 = (c^{3-1} - 1) - 1.$$

If neither a nor b is divisible by 3, then each of the terms on the left-hand side of